Math 261B Tues, Nov. 3 – Election Day

$U_{\mathbb{Z}}(\mathfrak{g})$ **Recall** – (char $k=0$) $\mathcal{U}(\mathfrak{g}) \hookrightarrow \mathcal{O}(G)^*$

$\qquad$ subalg of $\xi \in \mathcal{O}(G)^*$ s.t. some $m_e^N \subset \ker \xi$

$\to$ $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+ \implies \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_-) \otimes \mathcal{U}(\mathfrak{t}) \otimes \mathcal{U}(\mathfrak{n}_+)$

$\to$ $U_{\mathbb{Z}}(\mathfrak{n}_+)$ gen. by $E_i^{(m)}$ (by def'n, but

$\to$ $U_{\mathbb{Z}}(\mathfrak{n}_-)$ " " $F_i^{(m)}$ it also has $E_\alpha^{(m)}$ $\forall \alpha \in R_+$)

$\to$ $U_{\mathbb{Z}}(\mathfrak{t})$ is full $\mathbb{Z}$ dual in $\mathcal{U}(\mathfrak{t})$ of $\mathcal{O}_{\mathbb{Z}}(T) = \mathcal{O}(T)$

$$\langle \mathcal{U}(\mathfrak{t}), \mathcal{O}(T)\rangle \to K \qquad \mathcal{O}_{\mathbb{Z}}(T) = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

$$\langle U_{\mathbb{Z}}(\mathfrak{t}), \mathcal{O}_{\mathbb{Z}}(T)\rangle \to \mathbb{Z}$$

$$\mathcal{U}(\mathfrak{t}) = K[h_1, \dots, h_n] \qquad h_i = x_i \partial/\partial x_i$$

$U_{\mathbb{Z}}(\mathfrak{t})$ has basis $\qquad \langle h_i, x^\lambda\rangle = \lambda_i$

$$\left\{ \binom{h_1}{m_1} \cdots \binom{h_n}{m_n} \quad m_i \in \mathbb{N} \right\} \qquad \binom{x}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}$$

$$\to U_{\mathbb{Z}}(\mathfrak{g}) = \underbrace{U_{\mathbb{Z}}(\mathfrak{n}_-)}_{F_i\text{'s}} \otimes_{\mathbb{Z}} \underbrace{U_{\mathbb{Z}}(\mathfrak{t})}_{h_i\text{'s}} \otimes_{\mathbb{Z}} \underbrace{U_{\mathbb{Z}}(\mathfrak{n}_+)}_{E_i\text{'s}}$$

$F_1^{(k)} E_1^{(l)}$ $\quad$ $E_1^{(l)} F_1^{(k)}$ $\quad$ $E_1 F_1 = F_1 E_1 + [E_1, F_1]$, $\; [E_1,F_1] = H_1$

Why is this a subring?

$U_{\mathbb{Z}}(u_-)$

Straightening $\quad U_{\mathbb{Z}}(t)\, U_{\mathbb{Z}}(u_-)$ $\quad$ $f(h_1, \dots, h_n)\, w \overset{?}{=} \ldots \in U_{\mathbb{Z}}(u_-) \otimes U_{\mathbb{Z}}(t)$, $\; w \in U_{\mathbb{Z}}(u_-)$

$[h_i, w] = \lambda_i w$ $\quad$ $w \in U(u_-)$ has weight λ

$h_i w = \lambda_i w + w h_i$

$f(h_1, \dots, h_n)\, w$

T comodule V_λ, $\; h_i \circlearrowleft \lambda_i$

$U(g)$ is a Hopf algebra:

$x \in g$ are primitive -

$$\Delta x = x \otimes 1 + 1 \otimes x$$

$\xi \in O(G)^*$

$\xi(f_1 f_2) = \sum \xi_i(f_1)\, \xi_i'(f_2)$

$\Delta \xi = \sum \xi_i \otimes \xi_i'$

$f, w \in U(g)$

$\xi \circ \mu_{O(G)} \in O(G)^* \otimes O(G)^* \subset (O(G) \otimes O(G))^*$

$$f w = \sum (\mathrm{Ad} f_1)(w)\, f_2$$

$$\Delta f = \sum f_1 \otimes f_2$$

$$(\mathrm{Ad} f)\, w = \sum f_1\, w\, S(f_2)$$

Ex. 1) $e_g \in O(G)^*$

$e_g(f) = f(g)$ $\quad$ $e_g(f_1 f_2) = e_g(f_1)\, e_g(f_2)$

$\Delta e_g = e_g \otimes e_g$

$\sum f_1\, w\, \underbrace{S(f_2)\, f_3}$

$= \sum f_1\, w\, \varepsilon(f_2)$

$\parallel$

$f\, w$

$\sum f_1\, \varepsilon(f_2)$

$\parallel$

f

2) $x \in \mathfrak{g}$ $\quad x(f_1 f_2)$

$= x(f_1)\, f_2(e) + f_1(e)\, x(f_2)$

$\Delta x = x \otimes 1 + 1 \otimes x \quad \longleftarrow$

$1 \in \mathcal{O}(G)^*$ $\quad 1(f) = f(e)$

$1 = e_e$

$(\operatorname{Ad} h_i)\, w = [h_i, w] = \lambda_i w \quad h \otimes 1 + 1 \otimes h$

$h \otimes 1 + 1 \otimes S(h)$

$S(h) = -h$

$f(\underline{h})\, w =$

$\in U_{\mathbb{Z}}(t)$

$\sum f_1(\lambda)\, w\, f_2$

$f_1(\lambda) \in \mathbb{Z}$, $f_2 \in U_{\mathbb{Z}}(t)$

$(\operatorname{Ad} f(h_1, \dots, h_n))\, w = f(\lambda_1, \dots, \lambda_n)$

Need to know $U_{\mathbb{Z}}(t)$ is a Hopf subalg. of $U(t)$ — "obvious"

$\binom{h_1}{m_1} \cdots \binom{h_n}{m_n}$ $\quad \Delta \binom{h_i}{m} = \binom{h_i \otimes 1 + 1 \otimes h_i}{m} = \sum_{k+l=m} \binom{h_i}{k} \otimes \binom{h_i}{l}$

$\sum \binom{h}{m} t^m$ is is formally grouplike

$\parallel$

$(1+t)^h$

$\binom{x+y}{m} = \sum_{k+l=m} \binom{x}{k}\binom{y}{l}$

What about $E_i^{(k)} F_j^{(l)}$? If $i \neq j$, they commute.

$$= F_j^{(l)} E_i^{(k)}$$

" " $E_i^{(k)} F_i^{(l)}$

It's an $\mathfrak{sl}_2$ problem: $E^{(k)} F^{(l)} \overset{?}{\in} \mathbb{Z} \cdot \left\{ F^{(r)} \binom{H}{m} E^{(s)} \right\}$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \exp xE = \sum_k E^{(k)} x^k \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \exp yF = \sum_l F^{(l)} y^l \qquad 1 + xE$$

$$\cancel{\begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix} = \exp uH} \qquad \binom{H}{m} \qquad \begin{pmatrix} t & \\ & -t \end{pmatrix}$$

$$\begin{pmatrix} 1+t & 0 \\ 0 & (1+t)^{-1} \end{pmatrix} = \exp H \log(1+t) \qquad 1+t = e^u$$

$$= (1+t)^H = \sum \binom{H}{m} t^m$$

$$E^{(k)} F^{(l)} = \langle x^k y^l \rangle \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \langle x^k y^l \rangle \begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix}$$

L·D·U factor $\begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{y}{1+xy} & 1 \end{pmatrix} \begin{pmatrix} 1+xy & x \\ 0 & 1 - \frac{xy}{1+xy} \end{pmatrix}$

$$1 - \frac{xy}{1+xy} = \frac{1}{1+xy}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{y}{1+xy} & 1 \end{pmatrix} \begin{pmatrix} 1+xy & 0 \\ 0 & (1+xy)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{x}{1+xy} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1+xy & \frac{x}{1} \\ & \frac{1}{1+xy} \end{pmatrix} = \begin{pmatrix} 1+xy & \\ & \frac{1}{1+xy} \end{pmatrix}. \begin{pmatrix} 1 & \frac{x}{1+xy} \\ 0 & 1 \end{pmatrix}$$

$$= \left(\sum_r \left(\frac{y}{1+xy}\right)^r F^{(r)} \right) \left(\sum_m (1+xy)^m \binom{H}{m} \right) \left(\sum_s \left(\frac{x}{1+xy}\right)^s E^{(s)} \right)$$

Take coefficient of $x^k y^l$ $\quad \sum_{m,d} \cdot F^{(l-d)} \binom{H}{m} E^{(k-d)}$

$$\langle x^k y^l \rangle \quad y^{l-d} x^{k-d} (1+xy)^{m-(l-d)-(k-d)}$$

$$\langle (xy)^m \rangle$$

$$\binom{m-(l-d)-(k-d)}{m} = 0 \text{ if } m > (l-d)+(k-d)$$

$$E^{(k)} F^{(l)} = \sum_{m,d} \binom{m-(l-d)-(k-d)}{m} F^{(l-d)} \binom{H}{m} E^{(k-d)}.$$

$PSL_2 \cong SO_3$ over $\mathbb{Z}$?

defining rep

$$\mathcal{O}_{\mathbb{Z}}(SO_n) = \mathbb{Z}[a_{11}, \dots, a_{nn}] / \left(Q(Av) = Q(v),\ \det(A) = 1 \right)$$

$$Q(v) = x_1 x_n + x_2 x_{n-1} + \dots + \begin{cases} x_m x_m & n = 2m-1 \\ x_m x_{m+1} & n = 2m \end{cases}$$

$\mathcal{O}_{\mathbb{Z}}(SO_3)$:

$$V^2_{\mathbb{Z}} = \langle x^2, 2xy, y^2 \rangle$$

SL_2

x_1, x_2, x_3

a, b, c

$$v = ax^2 + 2bxy + cy^2$$

$$Q(v) = ac - b^2 = \text{discriminant}$$

$x_1 x_3 - x_2^2$

SL_2 invariant

$$\mathbb{Z}[a_{11}, \dots, a_{33}] / (Q(Av) = Q(v),\ \det(A) = 1)$$

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$V^2 \cong sl_2$$

$$x^2, 2xy, y^2 \leftrightarrow E, -H, -F$$

$$(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)(a_{31}x_1 + a_{32}x_2 + a_{33}x_3)$$
$$- (a_{21}x_1 + a_{22}x_2 + a_{23}x_3)^2 \leftarrow Q(Av)$$
$$= x_1 x_3 - x_2^2$$

$ac - b^2 \quad aE - bH - cF$

$$= \det \begin{pmatrix} -b & a \\ c & b \end{pmatrix}$$

$\langle x_1 x_3 \rangle$	$a_{11}a_{33} + a_{13}a_{31} - 2a_{21}a_{23}$	$=$	1
$\langle x_2 x_3 \rangle$	$a_{12}a_{33} + a_{13}a_{32} - 2a_{22}a_{23}$	$=$	0
$\langle x_1 x_2 \rangle$	$a_{11}a_{32} + a_{12}a_{31} - 2a_{21}a_{22}$	$=$	0
$\langle x_1^2 \rangle$	$a_{11}a_{31} - a_{21}^2$	$=$	0
$\langle x_2^2 \rangle$	$a_{12}a_{32} - a_{22}^2$	$=$	-1
$\langle x_3^2 \rangle$	$a_{13}a_{33} - a_{32}^2$	$=$	0

$(\varepsilon m_{13})(\varepsilon m_{31})$

$(1+\varepsilon m_{11})(1+\varepsilon m_{33})$

$= 1 + \varepsilon m_{11} + \varepsilon m_{33}$

$\det(A) = 1$

$A = I + \varepsilon M \qquad \varepsilon^2 = 0$

$\mathfrak{so}_3$: M s.t. $\mathrm{tr}\, M = 0$ $\Downarrow$ $m_{22} = 0$

$(SL_3$ vs. $GL_3)$

$m_{11} + m_{33} = 0$

$m_{12} - 2m_{23} = 0$

$m_{32} - 2m_{21} = 0$

$m_{31} = 0$

$(2m_{22} = 0)$

$m_{13} = 0$

$$M = \begin{pmatrix} x & 2y & 0 \\ z & 0 & y \\ 0 & 2z & -x \end{pmatrix}$$

$SO_3/\mathbb{Z}$ is a smooth group scheme over $\mathbb{Z}$ $\Rightarrow$

$\longrightarrow$ $\mathfrak{so}_3$ is free abelian group of rank 3